

On correspondences of a K3 surface with itself. III

C.G.Madonna ^{*} and Viacheslav V.Nikulin [†]

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Abstract

Let X be a K3 surface, and H its primitive polarization of the degree $H^2 = 2rs$, $r, s \geq 1$. The moduli space of sheaves over X with the isotropic Mukai vector (r, H, s) is again a K3 surface, Y . In [2], [3] and [9] (in general) we gave necessary and sufficient conditions in terms of Picard lattice $N(X)$ of X when Y is isomorphic to X , under the additional condition $H \cdot N(X) = \mathbf{Z}$.

Here we show that these conditions imply existence of an isomorphism between Y and X which is a composition of some universal isomorphisms between moduli of sheaves over X , and Tyurin's isomorphism between moduli of sheaves over X and X itself. It follows that for a general K3 surface X with $H \cdot N(X) = \mathbf{Z}$ and $Y \cong X$, there exists an isomorphism $Y \cong X$ which is a composition of the universal and the Tyurin's isomorphisms.

This generalizes our recent results [4] for $r = s = 2$ on similar subject.

1 Introduction

We consider algebraic K3 surfaces X over the field \mathbf{C} of complex numbers. For a Mukai vector $v = (r, c_1, s)$ where $r \in \mathbf{N}$, $s \in \mathbf{Z}$ and $c_1 \in N(X)$, Picard lattice of X , we denote by $Y = M_X(r, c_1, s)$ the moduli space of stable (with respect to some ample $H' \in N(X)$) rank r sheaves on X with first Chern classes c_1 , and Euler characteristic $r + s$. The general common divisor of the Mukai vector v is

$$(r, c_1, s) = (r, d, s)$$

if $c_1 = dc'_1$ where $d \in \mathbf{N}$ and c'_1 is primitive in the Picard lattice $N(X)$ which is a free \mathbf{Z} -module of the rank $\rho(X) = \text{rk } N(X)$. Here $c'_1 \in N(X)$ is *primitive* means that $c'_1/n \notin N(X)$ for any natural $n \geq 2$. A Mukai vector v is called *primitive* if its general common divisor is one.

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By results of Mukai [5], [6], under suitable conditions on the Chern classes, the moduli space Y is always deformations equivalent to a Hilbert scheme of 0-dimensional cycles on X (of the same dimension).

In [2] for $r = s = 2$, in [3] for $r = s = c \in \mathbf{N}$, and in [9] for arbitrary $r, s \in \mathbf{N}$, the following result had been obtained.

Theorem 1.1. *Let X be a K3 surface with a polarization H such that $H^2 = 2rs$, $r, s \geq 1$, the Mukai vector (r, H, s) be primitive, and $Y = M_X(r, H, s)$ be the moduli of sheaves on X with the Mukai vector (r, H, s) .*

Then $Y \cong X$ if at least for one of signs \pm there exists $h_1 \in N(X)$ such that the elements H and h_1 are contained in a 2-dimensional sublattice $N \subset N(X)$ with $H \cdot N = \mathbf{Z}$ and h_1 belongs to either the a -series or the b -series described below where $c = (r, s)$, $a = r/c$, $b = s/c$:

a-series:

$$h_1^2 = \pm 2bc, \quad H \cdot h_1 \equiv 0 \pmod{bc}, \quad (1.1)$$

b-series:

$$h_1^2 = \pm 2ac, \quad H \cdot h_1 \equiv 0 \pmod{ac}. \quad (1.2)$$

The conditions above are necessary for $H \cdot N(X) = \mathbf{Z}$ and $Y \cong X$ if $\rho(X) \leq 2$ and X is a general K3 surface with its Picard lattice, I. e., the automorphism group of the transcendental periods $(T(X), H^{2,0}(X))$ of X is ± 1 .

In formulation of this Theorem 1.1 in [9] some additional primitivity conditions on h_1 were also required. We will show in Sect. 2, Remark 2.1, that they are unnecessary.

The sufficient part of the proof of Theorem 2.1 in [9] and similar Theorems in [2], [3] used global Torelli Theorem for K3 surfaces [11]. I. e., under conditions of Theorem 2.1, we proved that the K3 surfaces X and Y have isomorphic periods. By global Torelli Theorem, then X and Y are isomorphic.

In Sect. 3 below we will give a geometric construction of the isomorphism between X and Y which is similar to our considerations in [4]. This is the main result of this paper.

We prove the following result.

Theorem 1.2. *Let X be a K3 surface with a polarization H such that $H^2 = 2rs$, $r, s \geq 1$, the Mukai vector (r, H, s) be primitive, and $Y = M_X(r, H, s)$ be the moduli of sheaves on X with the Mukai vector (r, H, s) .*

Assume that at least for one of signs \pm there exists $h_1 \in N(X)$ such that the elements H and h_1 are contained in a 2-dimensional sublattice $N \subset N(X)$ with $H \cdot N = \mathbf{Z}$ and h_1 belongs to either the a -series or the b -series described below where $c = (r, s)$, $a = r/c$, $b = s/c$:

a-series:

$$h_1^2 = \pm 2bc, \quad H \cdot h_1 \equiv 0 \pmod{bc};$$

b-series:

$$h_1^2 = \pm 2ac, \quad H \cdot h_1 \equiv 0 \pmod{ac}.$$

Then Y is isomorphic to X with the isomorphism given by the composition of the universal isomorphisms δ (if necessary), $\nu(1, d_2)$, T_D and $Tyu(\pm h_1)$ (see (3.6), (3.7)) which have very clear geometric meaning.

Here δ is the reflection $\delta : M_X(r, H, s) \cong M_X(s, H, r)$ (see [12], [13]), the isomorphism $\nu(d_1, d_2) : M_X(r, H, s) \cong M_X(d_1^2 r, d_1 d_2 H, d_2^2 s)$ where $d_1, d_2 \in \mathbf{N}$ and $(d_1, s) = (d_2, r) = (d_1, d_2) = 1$ are some universal isomorphisms of moduli of sheaves over an arbitrary algebraic K3 surface. The isomorphism T_D is defined by the tensor product with $\mathcal{O}(D)$ for a class of divisors $D \in N(X)$, thus it is also very universal. The isomorphism $Tyu(\pm h_1) : M_X(s, h_1, \pm 1) \cong X$ was geometrically defined and used by A.N. Tyurin [13] (see also [12] and [14]).

It follows a very interesting corollary, which shows that the universal isomorphisms δ , $\nu(d_1, d_2)$, T_D and Tyu are sufficient to find an isomorphism between $M_X(r, H, s)$ and X , if it does exist, for a general K3 surface X with $H \cdot N(X) = \mathbf{Z}$. Then there exists an isomorphism between $M_X(r, H, s)$ and X which is their composition.

Corollary 1.1. *Let X be a K3 surface with a polarization H such that $H^2 = 2rs$, $r, s \geq 1$, the Mukai vector (r, H, s) be primitive, and $Y = M_X(r, H, s)$ be the moduli of sheaves over X with the Mukai vector (r, H, s) .*

Then, if $Y \cong X$ and $H \cdot N(X) = \mathbf{Z}$ and X is general satisfying these properties (exactly here general X means that $\rho(X) = 2$ and the automorphism group of the transcendental periods $\text{Aut}(T(X), H^{2,0}(X)) = \pm 1$), there exists an isomorphism between Y and X which is a composition of the universal isomorphisms δ , $\nu(d_1, d_2)$ and T_D between moduli of sheaves over X , and the universal Tyurin's isomorphism Tyu between a moduli of sheaves over X and X itself.

We remark that here the isomorphisms δ , T_D and in general Tyu have a geometric description which does not use Global Torelli Theorem for K3 surfaces. Only for the isomorphism $\nu(d_1, d_2)$ we don't know a geometric construction. On the other hand, the isomorphism $\nu(d_1, d_2)$ is very universal and it exists even for general (with Picard number one) K3 surfaces. By [8], there exists only one isomorphism between general algebraic K3 surfaces (or two for degree two). Thus, we can consider this isomorphism as geometric by definition.

2 Reminding of the main results from [2], [3] and [10]

We denote by X an algebraic K3 surface over the field \mathbf{C} of complex numbers. I.e. X is a non-singular projective algebraic surface over \mathbf{C} with the trivial canonical class $K_X = 0$ and the vanishing irregularity $q(X) = 0$.

We denote by $N(X)$ the Picard lattice (i.e. the lattice of 2-dimensional algebraic cycles) of X . By $\rho(X) = \text{rk } N(X)$ we denote the Picard number of X . By

$$T(X) = N(X)_{H^2(X, \mathbf{Z})}^\perp \quad (2.1)$$

we denote the transcendental lattice of X .

For a Mukai vector $v = (r, c_1, s)$ where $r \in \mathbf{N}$, $s \in \mathbf{Z}$, and $c_1 \in N(X)$, we denote by $Y = M_X(r, c_1, s)$ the moduli space of stable (with respect to some ample $H' \in N(X)$) rank r sheaves on X with first Chern classes c_1 , and Euler characteristic $r + s$.

By results of Mukai [5], [6], under suitable conditions on the Chern classes, the moduli space Y is always deformations equivalent to a Hilbert scheme of 0-dimensional cycles on X (of same dimension).

In [2] for $r = s = 2$, in [3] for $r = s = c \in \mathbf{N}$, and in [9] for arbitrary $r, s \in \mathbf{N}$, the following result had been obtained.

Theorem 2.1. *Let X be a K3 surface with a polarization H such that $H^2 = 2rs$, $r, s \geq 1$, the Mukai vector (r, H, s) be primitive, and $Y = M_X(r, H, s)$ be the moduli of sheaves on X with the Mukai vector (r, H, s) .*

Then $Y \cong X$ if at least for one of signs \pm there exists $h_1 \in N(X)$ such that the elements H and h_1 are contained in a 2-dimensional sublattice $N \subset N(X)$ with $H \cdot N = \mathbf{Z}$ and h_1 belongs to either the a -series or the b -series described below where $c = (r, s)$, $a = r/c$, $b = s/c$:

a -series:

$$h_1^2 = \pm 2bc, \quad H \cdot h_1 \equiv 0 \pmod{bc}; \quad (2.2)$$

b -series:

$$h_1^2 = \pm 2ac, \quad H \cdot h_1 \equiv 0 \pmod{ac}. \quad (2.3)$$

The conditions above are necessary for $H \cdot N(X) = \mathbf{Z}$ and $Y \cong X$ if $\rho(X) \leq 2$ and X is a general K3 surface with its Picard lattice, I. e., the automorphism group of the transcendental periods $(T(X), H^{2,0}(X))$ of X is ± 1 .

Remark 2.1. In [9], in (2.2), (2.3) some additional primitivity conditions on h_1 were required. Thus, the corresponding conditions in [9] were for a -series:

$$h_1^2 = \pm 2bc, \quad H \cdot h_1 \equiv 0 \pmod{bc}, \quad H \cdot h_1 \not\equiv 0 \pmod{bcl_1}, \quad h_1/l_2 \notin N(X), \quad (2.4)$$

where l_1 and l_2 are any primes such that $l_1^2|a$ and $l_2^2|b$;

for b -series:

$$h_1^2 = \pm 2ac, \quad H \cdot h_1 \equiv 0 \pmod{ac}, \quad H \cdot h_1 \not\equiv 0 \pmod{acl_1}, \quad h_1/l_2 \notin N(X), \quad (2.5)$$

where l_1 and l_2 are any primes such that $l_1^2|b$ and $l_2^2|a$.

Let us show that actually (2.2) is equivalent to (2.4), and (2.3) is equivalent to (2.5) (where we assume that $H \cdot N(X) = \mathbf{Z}$). Thus, in sufficient part of Theorem 2.1 it is better to use (2.2), (2.3), and in necessary part of Theorem 2.1 it is better to use (2.4), (2.5).

According to [9], for some $\mu \pmod{2abc^2} \in (\mathbf{Z}/2abc^2)^*$ and $d \in \mathbf{N}$ such that $\mu^2 \equiv d \pmod{4abc^2}$, (2.2) is equivalent to the system of diophantine conditions (see (3.54) in [9])

$$p^2 - dq^2 = \pm 4ac, \quad p \equiv \mu q \pmod{2ac}, \quad (2.6)$$

and (2.4) is equivalent to

$$p^2 - dq^2 = \pm 4ac, \quad p \equiv \mu q \pmod{2ac}, \quad (a, p, q) = 1, \quad p \not\equiv \mu q \pmod{2asl} \quad (2.7)$$

for any prime $l|b$.

Let us show that (2.6) implies (2.7). We can find $\mu_0 \pmod{2a^2c^2}$ such that $\mu_0 \equiv \mu \pmod{2ac^2}$ and $d \equiv \mu_0^2 \pmod{4a^2c^2}$. We then get $p^2 - dq^2 \equiv (p - \mu_0 q)(p + \mu_0 q) \equiv \pm 4ac \pmod{4a^2c^2}$. Then

$$\frac{p - \mu_0 q}{2ac} \cdot (p + \mu_0 q) \equiv \pm 2 \pmod{2ac}$$

where $(p - \mu_0 q)/(2ac)$ is an integer. This leads to a contradiction if a prime $l|(a, p, q)$.

Similarly, since $\mu^2 \equiv d \pmod{4abc^2}$, we obtain that

$$\frac{p - \mu q}{2ac} \cdot \frac{p + \mu q}{2} \equiv \pm 1 \pmod{bc}.$$

This leads to a contradiction if $p - \mu q \equiv 0 \pmod{2acl}$ for a prime $l|b$.

Thus, (2.6) implies (2.7), and they are equivalent indeed.

The sufficient part of the proof of Theorem 2.1 in [9] and similar Theorems in [2], [3] used global Torelli Theorem for K3 surfaces [11]. I. e., under conditions of Theorem 2.1, we proved that the K3 surfaces X and Y have isomorphic periods. By global Torelli Theorem, then X and Y are isomorphic.

In Sect. 3 below we will give a geometric construction of the isomorphism between X and Y which is similar to our considerations in [4]. This is the main result of this paper.

3 Geometric interpretation of the main results from [2], [3] and [10]

We use notations of the previous Section 2.

We shall use the natural isomorphisms between moduli spaces of sheaves over a K3 surface X .

Lemma 3.1. *Let (r, H, s) be a primitive Mukai vector for a K3 surface X and $r, s \geq 1$. Then one has an isomorphism, called **reflection**, see [13],*

$$\delta : M_X(r, H, s) \cong M_X(s, H, r).$$

In ([12], (4.11)) and [13], Lemma 3.4, the geometric construction of the reflection δ is given under the condition that $M_X(r, H, s)$ contains a regular bundle. On the other hand, by global Torelli Theorem for K3 surfaces [11], existence of such isomorphism is obvious. See similar proof of Theorem 3.1 below.

Lemma 3.2. *Let (r, H, s) be a Mukai vector for a K3 surface X and $D \in N(X)$. Then one has the natural isomorphism given by the tensor product*

$$T_D : M_X(r, H, s) \cong M_X(r, H + rD, s + r(D^2/2) + D \cdot H), \quad \mathcal{E} \mapsto \mathcal{E} \otimes \mathcal{O}(D).$$

Moreover, here Mukai vectors

$$v = (r, H, s), \quad v_1 = (r, H + rD, s + r(D^2/2) + D \cdot H)$$

have the same general common divisor and the same square under Mukai pairing. In particular, they are primitive and isotropic simultaneously.

We also use the isomorphisms between moduli of sheaves over a K3 surface and the K3 surface itself found by A.N.Tyurin in ([13], Lemma 3.3).

Lemma 3.3. *For a K3 surface X and $h_1 \in N(X)$ such that $\pm h_1^2 > 0$ and*

$$h^0 \mathcal{O}(h_1) = h^0 \mathcal{O}(-h_1) = 0 \text{ if } h_1^2 < 0, \quad (3.1)$$

there is a geometric Tyurin's isomorphism

$$Tyu(\pm h_1) : M_X(\pm h_1^2/2, h_1, \pm 1) \cong X.$$

Like in Theorem 3.1 below, using global Torelli Theorem for K3 surfaces [11], one can show that even without Tyurin's condition (3.1) always there exists some isomorphism

$$M_X(\pm h_1^2/2, h_1, \pm 1) \cong X.$$

We call such isomorphisms also as *Tyurin's isomorphisms*. When h_1 also satisfies (3.1), there exists a direct Tyurin's geometric construction of some isomorphism $M_X(\pm h_1^2/2, h_1, \pm 1) \cong X$.

We shall use the following result which had been proved implicitly in [10]. See Sect. 2.3 and the proof of Theorem 2.3.3 in [10]. In [10] much more difficult results related to arbitrary Picard lattice had been considered, and respectively the proofs were long and difficult. Therefore, below we also give a much simpler proof of this result.

Theorem 3.1. *Let $v = (r, H, s)$ be an isotropic Mukai vector on a K3 surface X where $r, s \in \mathbf{N}$, $H \in N(X)$, $H^2 = 2rs$, and H is primitive (then v is also primitive).*

Let $d_1, d_2 \in \mathbf{N}$ and $(d_1, s) = (d_2, r) = (d_1, d_2) = 1$.

Then the Mukai vector $v_1 = (d_1^2 r, d_1 d_2 H, d_2^2 s)$ is also primitive, and there exists a natural isomorphism of moduli of sheaves

$$\nu(d_1, d_2) : M_X(r, H, s) \cong M_X(d_1^2 r, d_1 d_2 H, d_2^2 s).$$

Proof. Let us consider the case $(d_1, d_2) = (d, 1)$ where $(d, s) = 1$ (general case is similar).

Let $c = (r, s)$ and $a = r/c$, $b = s/c$. Then $v = (ac, H, bc)$ where $H^2 = 2abc^2$ and H is primitive. By global Torelli Theorem [11], it is enough to show that periods of $Y = M_X(r, H, s)$ and $Y_1 = M_X(d^2r, dH, s)$ are isomorphic.

By results of Mukai [5], [6], cohomology of Y (and similarly of Y_1) are equal to

$$H^2(Y, \mathbf{Z}) = v^\perp / \mathbf{Z}v$$

where we consider v as the element of Mukai lattice $H^*(X, \mathbf{Z})$, and $H^{2,0}(Y)$ is the image of $H^{2,0}(X)$.

Using the variant of Witt's Theorem from [11], we can model the necessary calculations as follows.

Let $U^{(1)}$ be an even unimodular hyperbolic plane with the basis e_1, e_2 where $e_1^2 = e_2^2 = 0$ and $e_1 \cdot e_2 = -1$. Let $U^{(2)}$ be another even unimodular hyperbolic plane with the bases f_1, f_2 where $f_1^2 = f_2^2 = 0$ and $f_1 \cdot f_2 = 1$. We consider the orthogonal sum $U^{(1)} \oplus U^{(2)}$ (the model of $H^*(X, \mathbf{Z})$; to get $H^*(X, \mathbf{Z})$, one should add to $U^{(1)} \oplus U^{(2)}$ an unimodular even lattice of signature $(2, 18)$ which is the same for all three X, Y, Y_1). Then

$$v = ace_1 + bce_2 + H, \quad H = abc^2 f_1 + f_2,$$

thus

$$v = ace_1 + bce_2 + abc^2 f_1 + f_2.$$

Then $N(X) = \mathbf{Z}H$ models the Picard lattice of X and $T(X) = \mathbf{Z}t$, $t = -abc^2 f_1 + f_2$ models the transcendental lattice of X .

We have $\xi = x e_1 + y e_2 + z f_1 + w f_2 \in v^\perp$ if and only if $-bcx - acy + z + abc^2 w = 0$, equivalently $z = bcx + acy - abc^2 w$ where $x, y, w, z \in \mathbf{Z}$. Thus,

$$\xi = x(e_1 + bc f_1) + y(e_2 + ac f_1) + w(-abc^2 f_1 + f_2)$$

and $\alpha = e_1 + bc f_1$, $\beta = e_2 + ac f_1$, $t = -abc^2 f_1 + f_2$ give the basis of v^\perp . We have $v = ac \alpha + bc \beta + t$ and

$$ac \alpha \bmod \mathbf{Z}v + bc \beta \bmod \mathbf{Z}v + t \bmod \mathbf{Z}v = 0.$$

It follows that $\bar{\alpha} = \alpha \bmod \mathbf{Z}v$, $\bar{\beta} = \beta \bmod \mathbf{Z}v$ give a basis of $v^\perp / \mathbf{Z}v$ which models $H^2(Y, \mathbf{Z})$. We have $\bar{\alpha}^2 = \bar{\beta}^2 = 0$ and $\bar{\alpha} \cdot \bar{\beta} = -1$. Thus $v^\perp / \mathbf{Z}v \cong U$. We see that $\tilde{t} = t \bmod \mathbf{Z}v = -ac\bar{\alpha} - bc\bar{\beta}$ and then $\tilde{t} = \tilde{t}/c = -a\bar{\alpha} - b\bar{\beta} \in (v^\perp / \mathbf{Z}v)$, and $\mathbf{Z}\tilde{t}$ models the transcendental lattice $T(Y)$ of Y . Its orthogonal complement $\mathbf{Z}h$ where $h = a\bar{\alpha} - b\bar{\beta}$ models the Picard lattice $N(Y)$ of Y . We have $h^2 = 2ab > 0$.

Let us make similar calculations for Y_1 . We have

$$v_1 = d^2 ac e_1 + bc e_2 + dH, \quad H = abc^2 f_1 + f_2,$$

and

$$v_1 = d^2 ac e_1 + bc e_2 + dabc^2 f_1 + d f_2.$$

We have $\xi_1 = x_1 e_1 + y_1 e_2 + z_1 f_1 + w_1 f_2 \in v_1^\perp$ if and only if $-bcx_1 - d^2acy_1 + dz_1 + dabc^2w_1 = 0$. Since $(d, bc) = 1$, we obtain $x_1 = d\tilde{x}_1$, and $z_1 = bc\tilde{x}_1 + dacy_1 - abc^2w_1$ where $\tilde{x}_1, y_1, w_1, z_1 \in \mathbf{Z}$. Then

$$\xi_1 = \tilde{x}_1(de_1 + bcf_1) + y_1(e_2 + dacf_1) + w_1(-abc^2f_1 + f_2)$$

and $\alpha_1 = de_1 + bcf_1$, $\beta_1 = e_2 + dacf_1$, $t = -abc^2f_1 + f_2$ give the basis of v_1^\perp . We have $v_1 = dac\alpha_1 + bc\beta_1 + dt$, and

$$dac\alpha_1 \bmod \mathbf{Z}v_1 + bc\beta_1 \bmod \mathbf{Z}v_1 + dt \bmod \mathbf{Z}v_1 = 0.$$

Since $(d, bc) = 1$, we see that $t \bmod \mathbf{Z}v_1 = \tilde{c}\tilde{t}_1$, $\beta_1 \bmod \mathbf{Z}v_1 = d\tilde{\beta}_1$ where $\tilde{t}_1, \tilde{\beta}_1 \in v_1^\perp/\mathbf{Z}v_1$, and

$$a\alpha_1 \bmod \mathbf{Z}v_1 + b\tilde{\beta}_1 + \tilde{t}_1 = 0.$$

We have $(\alpha_1 \bmod \mathbf{Z}v_1)^2 = \tilde{\beta}_1^2 = 0$ and $(\alpha_1 \bmod \mathbf{Z}v_1) \cdot \tilde{\beta}_1 = -1$. Thus, again $\alpha_1 \bmod \mathbf{Z}v_1$, $\tilde{\beta}_1$ give canonical generators of the unimodular lattice U . Then they give a basis of $v_1^\perp/\mathbf{Z}v_1$ which models $H^2(Y_1)$. Moreover $\mathbf{Z}\tilde{t}_1$ where $\tilde{t}_1 = -a\alpha_1 \bmod \mathbf{Z}v_1 - b\tilde{\beta}_1$ (from above), models the transcendental lattice of Y_1 . Its orthogonal complement $\mathbf{Z}h_1$ where $h_1 = a\alpha_1 \bmod \mathbf{Z}v_1 - b\tilde{\beta}_1$ models the Picard lattice of Y_1 .

We see that our descriptions for Y and Y_1 above are evidently isomorphic if we identify $\bar{\alpha}, \bar{\beta}$ with $\alpha_1 \bmod \mathbf{Z}v$ and $\tilde{\beta}_1$ respectively. This shows that Y and Y_1 have isomorphic periods and are isomorphic by global Torelli Theorem for K3 surfaces [11].

This finishes the proof. \square

Remark 3.1. It would be very interesting to find a direct geometric proof of Theorem 3.1 which does not use global Torelli Theorem for K3 surfaces. It seems, considerations by Mukai in [7] are related with this problem (especially see Theorem 1.2, in [7]).

On the other hand, the isomorphism $\nu(d_1, d_2)$ is very universal, and it exists even for general (with Picard number one) K3 surfaces. By [8], there exists only one isomorphism (or two isomorphisms for the degree two) between algebraic K3 surfaces with Picard number one. Thus, we can consider this isomorphism as geometric by definition.

We will show below that under the conditions of Theorem 2.1, there exists an isomorphism between X and Y which is the composition of the universal geometric isomorphisms above.

Theorem 3.2. *Let X be a K3 surface with a polarization H such that $H^2 = 2rs$, $r, s \geq 1$, the Mukai vector (r, H, s) be primitive, and $Y = M_X(r, H, s)$ be the moduli of sheaves on X with the Mukai vector (r, H, s) .*

Assume that at least for one of signs \pm there exists $h_1 \in N(X)$ such that the elements H and h_1 are contained in a 2-dimensional sublattice $N \subset N(X)$ with $H \cdot N = \mathbf{Z}$ and h_1 belongs to either the a -series or the b -series described below where $c = (r, s)$, $a = r/c$, $b = s/c$:

a-series:

$$h_1^2 = \pm 2bc, \quad H \cdot h_1 \equiv 0 \pmod{bc};$$

b-series:

$$h_1^2 = \pm 2ac, \quad H \cdot h_1 \equiv 0 \pmod{ac}.$$

Then Y is isomorphic to X with the isomorphism given by the composition of the reflection δ (if h_1 belongs to the a -series), $\nu(1, d_2)$ (for some d_2), T_D (for some $D \in N$) and $T_{yu}(\pm h_1)$ (see (3.6), (3.7)) where d_2 and D are defined in the proof below.

Proof. We use the description of the pair $H \in N$ given in Proposition 3.1 in [10] which directly follows from $\text{rk } N = 2$ and $H \cdot N = \mathbf{Z}$. We denote $d = -\det N$ and $\mathbf{Z}\delta$ the orthogonal complement to H in N . Then $\delta^2 = -2abc^2d$.

We have

$$N = [H, \delta, w = \frac{\mu H + \delta}{2abc^2}], \quad \mu \pmod{2abc^2} \in (\mathbf{Z}/2abc^2)^*, \quad \mu^2 \equiv d \pmod{4abc^2} \quad (3.2)$$

where $[\cdot]$ means ‘generated by \cdot ’. Here d and $\pm \mu \pmod{2abc^2} \subset (\mathbf{Z}/2abc^2)^*$ give the complete invariants of the pair $H \in N$ up to isomorphisms.

It follows that

$$N = \{z = \frac{xH + y\delta}{2abc^2} \mid x, y \in \mathbf{Z} \text{ and } x \equiv \mu y \pmod{2abc^2}\} \quad (3.3)$$

where

$$z^2 = \frac{x^2 - dy^2}{2abc^2}. \quad (3.4)$$

In the conditions of Theorem 3.1, let us assume that h_1 belongs to the a -series.

Then $h_1 = (rH + s\delta)/(2abc^2)$ where $r, s \in \mathbf{Z}$ and $r \equiv \mu s \pmod{2abc^2}$. Since $h_1^2 = \pm 2bc$, it follows that $r^2 - ds^2 = \pm(2bc)(2abc^2)$. We have $H \cdot h_1 = r \equiv 0 \pmod{bc}$. Since $r \equiv \mu s \pmod{2abc^2}$, it follows $s \equiv 0 \pmod{bc}$. Denoting $r = pbc$ and $s = qbc$ where $p, q \in \mathbf{Z}$, we get that $h_1 = (pH + q\delta)/(2ac)$ where $p \equiv \mu q \pmod{2ac}$ and $p^2 - dq^2 = \pm 4ac$.

We have $\delta = 2abc^2w - \mu H$. Then

$$h_1 = \frac{pH + q(2abc^2w - \mu H)}{2ac} = \frac{p - \mu q}{2ac}H + qbcw \equiv \frac{p - \mu q}{2ac}H \pmod{bcN}.$$

Since $p \equiv \mu q \pmod{2ac}$, there exists $d_2 \in \mathbf{N}$ such that

$$d_2 \equiv \frac{p - \mu q}{2ac} \pmod{bc}. \quad (3.5)$$

Then

$$h_1 = d_2H + bcD, \quad D \in N.$$

Then (see Lemma 3.1)

$$\delta : (ac, H, bc) \rightarrow (bc, H, ac),$$

and (see Lemma 3.2)

$$T_D : (bc, d_2H, d_2^2ac) \rightarrow (bc, h_1, \pm 1)$$

since (bc, d_2H, d_2^2ac) is isotropic Mukai vector and $h_1^2 = \pm 2bc$. Since $(bc, h_1, \pm 1)$ is evidently primitive, (bc, d_2H, d_2^2ac) is also primitive, and then $(d_2, bc) = 1$. By Theorem 3.1, then

$$\nu(1, d_2) : M_X(bc, H, ac) \cong M_X(bc, d_2H, d_2^2ac)$$

and

$$T_D : M_X(bc, d_2H, d_2^2ac) \cong M_X(bc, h_1, \pm 1).$$

At last (see Lemma 3.3),

$$Tyu(\pm h_1) : M_X(bc, h_1, \pm 1) \cong X.$$

Thus, we obtain the natural isomorphism

$$Tyu(\pm h_1) \cdot T_D \cdot \nu(1, d_2) \cdot \delta : M_X(r, H, s) \cong X. \quad (3.6)$$

If h_1 belongs to the b -series, similarly we can show that $h_1 = d_2H + acD$ for $d_2 \in \mathbf{N}$ and $D \in N$, and we obtain the natural isomorphism

$$Tyu(\pm h_1) \cdot T_D \cdot \nu(1, d_2) : M_X(r, H, s) \cong X. \quad (3.7)$$

We don't need the reflection δ in this case.

This finishes the proof of Theorem 3.2. \square

Since conditions of Theorem 3.2 are also necessary for a general K3 surface X with $H \cdot N(X) = \mathbf{Z}$ and $Y \cong X$ (see Theorem 2.1), we also obtain an interesting

Corollary 3.1. *Let X be a K3 surface with a polarization H such that $H^2 = 2rs$, $r, s \geq 1$, the Mukai vector (r, H, s) be primitive, and $Y = M_X(r, H, s)$ be the moduli of sheaves over X with the Mukai vector (r, H, s) .*

Then, if $Y \cong X$ and $H \cdot N(X) = \mathbf{Z}$ and X is general satisfying these properties (exactly here general X means that $\rho(X) = 2$ and the automorphism group of the transcendental periods $\text{Aut}(T(X), H^{2,0}(X)) = \pm 1$), there exists an isomorphism between Y and X which is a composition of the universal isomorphisms δ , $\nu(d_1, d_2)$ and T_D between moduli of sheaves over X , and the universal Tyurin's isomorphism Tyu between the moduli of sheaves over X and X itself.

Here it is important that the isomorphisms δ , T_D and Tyu have a geometric description which does not use Global Torelli Theorem for K3 surfaces. Only for the isomorphism $\nu(d_1, d_2)$ we don't know a geometric construction. On the other hand, the isomorphism $\nu(d_1, d_2)$ is very universal and is geometric by definition. See our considerations at the beginning of the Section.

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C.G.Madonna
Math. Dept., CSIC, C/ Serrano 121, 28006 Madrid, SPAIN
carlo@madonna.rm.it cgm@imaff.cfmac.csic.es

V.V. Nikulin
Deptm. of Pure Mathem. The University of Liverpool, Liverpool
L69 3BX, UK;
Steklov Mathematical Institute,
ul. Gubkina 8, Moscow 117966, GSP-1, Russia
vnikulin@liv.ac.uk vvnikulin@list.ru